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Asymptotic solutions for the distribution function in non-equilibrium flows. Part 1. The weak shock

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This paper is the first part of an investigation of the molecular velocity distribution function in non-equilibrium flows. In this part, the general features of the distribution are discussed and illustrated by a detailed study of its asymptotic expansions in different velocity domains for a weak shock, employing a simple relaxation model for the collisions. Using the strength of the shock as a small parameter, the Chapman–Enskog distribution is derived, in a less restrictive way than in previous analyses, as the first two terms in a suitable 'inner' asymptotic expansion of the distribution in velocity space, valid only for not very high velocities. It is shown that the consideration of two further limits, called the intermediate and outer, is necessary for a complete description of the distribution in velocity space. The uniformly valid composite expansion demonstrates the slow approach to equilibrium of fast molecules. The outer solution depends on integrals over the flow and is in general 'global', in contrast to the inner solution which is essentially local; this introduces certain asymmetries on a fine scale even in a weak shock. It is shown, for example, that fast molecules moving towards the hot side accumulate by collisionless streaming, whereas those moving towards the cold side attenuate like a molecular beam and represent essentially a 'precursor' of the hot side. A simple approximation for the distribution in the precursor is derived, and found to contain, in the outer limit, a large perturbation on the local Maxwellian; this results in an approach to equilibrium like $\exp(-|x|^{\frac{2}{3}})$ on the cold side.

A heuristic extension of the argument to the true Boltzmann equation leads to the result that for molecules with an interparticle potential varying as the inverse m-power of the distance, the approach to equilibrium through the precursor is like $\exp(-|x|^l)$, where $l = m/(m+2)$.

1. Introduction

The fundamental unknown in the kinetic theory of gases—or, as a matter of fact, in any gasdynamic problem—is the distribution function, whose development in space and time is governed by the Boltzmann equation. Because of the complexity and non-linearity of this equation, a detailed knowledge of the distribution is hard to achieve except in the limiting cases of complete equilibrium and collisionless flow. However, one is most often interested in only the first few moments of the distribution rather than in its detailed structure; and this motivation had led to the development of various methods which, in some sense, tend to focus attention on the moments.

In a rather special sense, the Chapman–Enskog–Hilbert theory of the Boltzmann equation (see Grad 1958 for an excellent discussion) was one of the earliest attempts to solve the equation in terms of the moments. Hilbert's analysis, extended by Grad (1958), showed that if the distribution f could be expanded in a certain small parameter, then it was a functional of only a few moments. This expansion procedure for f was expected at one time to yield the general solution of the Boltzmann equation (Chapman & Cowling 1952, p. ix), but it seems unable to do this as no satisfactory account can be taken of initial or boundary conditions, which logically can be prescribed only on f, and must therefore appear in any solution of the problem which is uniformly valid in space and time.

A different kind of approach, proposed and used by Grad (1949), assumes that the form of the distribution function in the space of molecular velocity $\mathbf{v}$ is the same as in a first approximation in the Chapman–Enskog solution, but leaves free the parameters in it, which are essentially the first thirteen moments of the distribution. Partial differential equations for these thirteen moments are then derived, and a solution of these equations was expected to give a more satisfactory description than the Chapman–Enskog approximations.

However, many difficulties (clearly summarized by Schaaf & Chambré 1958) soon appeared both with the higher (or Burnett) approximations in the Chapman–Enskog theory and with the thirteen-moment method of Grad, especially in flows with large departure from equilibrium. A third class of moment methods, designed to cover especially these highly non-equilibrium regimes of flow, are those proposed by Mott-Smith (1951) and Lees (1959) and extended and modified by various workers. In general, these methods again assume a certain form for the distribution function, departing most significantly from previous work in allowing for bimodal and discontinuous behaviour in velocity space. The assumed functions contain a few free parameters which are later evaluated by some kind of moment equation.

It is possible that eventually most problems in rarefied gasdynamics will have to be solved by some kind of moment technique. However, for a moment method to be useful, one needs to know the form of the distribution function reasonably well. Unfortunately, one's intuition about the distribution function does not yet seem strong enough to make reasonable guesses about its form. A major purpose of this report is to set forth some considerations which determine the structure of the distribution, and in particular to illustrate them by a fairly detailed study of the distribution function for a weak shock. The shock is an effective testing ground for various ideas and approximations, because it represents a highly non-equilibrium flow which has a simple geometry and is physically realizable. We consider the weak shock in detail because this provides a natural small parameter for studying asymptotic solutions. Further, we assume a simple relaxation model for the collision terms. This enables us to keep the discussion quite rigorous. However, many of the conclusions we reach here turn out to be true for strong shocks and other highly non-equilibrium flows, and are in fact capable of generalization to the true Boltzmann equation. Part 2 of this study will be devoted to some of these extensions, but one such result will be mentioned in §8 of this paper.

Before we proceed to the main analysis, some further remarks on the nature of the problem considered here and the results obtained may be useful at this stage. It is well known that in flow past bodies or in time-dependent flows, certain non-uniformities in the continuum solutions may be expected even when the departure from equilibrium is nominally small. For example, in flow past a body at small Knudsen numbers, the Navier–Stokes solution describes adequately most of the flow, but fails at distances of the order of a mean free path or less from the surface. At first sight, one would expect that such non-uniformities should not arise in steady flow in the absence of any surfaces, especially if, as in a weak shock, the (nominal) departure from equilibrium is small. One of the conclusions of the present work, however, is that there is a different and more basic non-uniformity in the continuum theories that makes such an expectation false.

Briefly, the reason for this is that the Chapman–Enskog solutions for the distribution, which lead to the Navier–Stokes and other continuum equations, are not uniformly valid in the space of molecular velocities (which we will call $\mathbf{v}$-space). This non-uniformity is well known, and is evident from the fact that at sufficiently large velocities the Chapman–Enskog perturbation term is no longer small, and, in fact, can even make the distribution negative. However, at large velocities the distribution function itself is usually very small, so it has been generally and tacitly assumed that the non-uniformity is not very important.

To test this assumption, we seek in the following analysis solutions for the distribution which are uniformly valid in $\mathbf{v}$-space. We freely use the terminology of the method of matched asymptotic expansions, although, as we are dealing essentially with integral rather than differential equations, the nature of the problem is rather different. From the view point of this technique, it will be shown that the Chapman–Enskog theory is only an inner expansion for the distribution in $\mathbf{v}$-space not valid for relatively high velocities. This inner expansion depends only on the 'local' flow and yields similarly local results (such as, for example, that the stress at any point is linear in the velocity gradient at the same point, and that the flow is determined by partial differential equations). However, to obtain a complete description of the distribution, two other limiting solutions are necessary. The outer solution, for example, will be found in general (but not necessarily always) to depend on the whole flow, and can be found only as the solution of integral equations; i.e. it is global in general. Only if the outer solution is not a large perturbation on local equilibrium (in some sense) will the Navier–Stokes solution be a good approximation to the flow. But even in a weak shock, there are regions (on the cold side, as we shall see) where the perturbation due to the outer solution dominates over the inner, and leads to an approach to equilibrium quite different from that given by the Navier–Stokes or any other continuum theory. A 'small' departure from equilibrium is thus only a necessary but not sufficient condition for the validity of the continuum theories.

At this point, the present analysis supports and adds to a very significant and interesting result obtained by Lyubarskii (1961), who has shown (with the model

Boltzmann equation) that the final approach to equilibrium in a weak shock is like $\exp(-C_1|x|^{\frac{2}{3}})$, rather than like $\exp(-C_2|x|)$, as would be expected by any of the continuum theories (C_1 and C_2 being some constants). Lyubarskii points out that the fast molecules are responsible for this result. His work involves an analysis of the asymptotic solutions of the integral equations for the moments, and avoids a discussion of the distribution itself. The present study, on the other hand, concentrates on the distribution (one of our aims being to understand its structure in some detail, possibly as a guide in the use of moment methods); but it so happens that once the two-sided nature of the distribution is grasped, one can derive Lyubarskii's result very simply by the use of the 'precursor' distribution, obtained in § 7. Moreover, the discussion of the distribution brings out explicitly an important distinction between molecules from the hotter and colder regions of the flow, and shows that the approach to equilibrium is like

$$\exp(-C_1|x|^{\frac{2}{3}})$$

only on the cold side of the shock. Finally, the simple derivation of the precursor distribution makes possible the immediate extension of the idea to other flows (involving the presence of surfaces and large departures from equilibrium), and to the true Boltzmann equation.

The plan of this paper is as follows: in § 2, we set forth some general considerations on the structure of the distribution, without reference to any particular flow or special molecular model. A brief resumé of the classical Chapman–Enskog approach and the Navier–Stokes solutions for the weak shock is given in § 3, as they are required later. The basic integral equation is formulated in § 4. The inner solution to the distribution is obtained in § 5, where its relation to the Enskog procedure is also discussed. The next two sections deal with the distribution at very high velocities, for molecules moving towards and away from the hot side, respectively. The precursor distribution and its consequences are discussed in § 7, and the results are extended to more general molecular models in § 8.

A brief, preliminary account of a part of §§ 5, 6 was presented at the Fifth International Symposium on Rarefied Gas Dynamics, held at Oxford in July 1966 (Narasimha 1967).

2. Some general considerations

The discussion in the rest of this report, excepting § 8, concerns a model of the Boltzmann equation. The purpose of this section is to present a general perspective which is independent of any particular collision model, but which nevertheless provides the motivation for the analysis of later sections. In particular, for flows with a small departure from equilibrium (such as that through a weak shock), we propose to obtain the Chapman–Enskog expansion by an asymptotic analysis which brings out the nature of the underlying assumptions.

We denote the distribution function by $f = f(\mathbf{x}, t; \mathbf{v})$ as usual; $\mathbf{x}$ is position, t the time and $\mathbf{v}$ the molecular velocity. We write the Boltzmann equation as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x}, t; \mathbf{v}) = G(f) - fL(f), \tag{2.1}$$

where G and L are the well-known gain and loss operators on f,

$$G(f) \equiv G(f,f) = \int f(\mathbf{v}')f(\mathbf{w}')gI\,d\Omega\,D\mathbf{w},$$

$$L(f) = \int f(\mathbf{w})gI\,d\Omega\,D\mathbf{w}.$$

Here g is the relative velocity between two molecules having velocities $\mathbf{v}$, $\mathbf{w}$ before collision and $\mathbf{v}'$, $\mathbf{w}'$ after collision; I is the cross-section for scattering into the elementary solid angle $d\Omega$; and $D\mathbf{w}$ is an element of volume in $\mathbf{w}$-space. We will write the gain term $G(f_1,f_2)$ if it involves two different distributions, as G is quadratic in f.

For many purposes it is convenient to rewrite (2.1) in the form

$$\frac{dt}{ds} = 1, \quad \frac{d\mathbf{x}}{ds} = \mathbf{v}, \quad \frac{df}{ds} + fL(f) = G(f), \tag{2.2}$$

where s is a parameter along the trajectory or the characteristic, equal to the time of flight at velocity $\mathbf{v}$. The integration of (2.2) gives

$$f(\sigma) = f(0)\,e^{-\sigma} + \int_0^\sigma \frac{G(\sigma')}{L(\sigma')} e^{-(\sigma-\sigma')}d\sigma', \tag{2.3}$$

where

$$\sigma = \int_0^s L(s')\,ds', \tag{2.4}$$

and $f(0)$ is an initial or boundary condition at $s = s_0$. (The fact that in flow past a body it may not be possible to prescribe $f(0)$ arbitrarily is not of much concern for our considerations here.) Enskog (1928) seems to have been the first to write the Boltzmann equation in this form. In general, when the molecules are considered as centres of force, the operators L and G will diverge, and it is assumed that some kind of cut-off, e.g. in scattering angle, has been applied before writing down (2.3). The first term in this equation represents the exponential decay of the initial condition (L evidently being of the order of the inverse of the mean free time t_c); the second term accounts for the net gain of particles over the characteristic, after allowing for attenuation by collision.

Now, if σ is very large, and G/L varies slowly in σ, the dominant contribution to $f(\sigma)$ may be estimated by an asymptotic evaluation of the integral in (2.3). Suppose, to be specific, that the flow variations (and hence f, $G(f)$ etc.) have a time scale $t_f \gg t_c$. Then, introducing

$$\theta \equiv t_c/t_f \ll 1, \quad \bar{s} = s/t_f, \tag{2.5}$$

we obtain the leading terms in an asymptotic expansion of (2.3) as

$$f(\bar{s}) = \frac{G(f)}{L(f)} - \frac{\theta}{Lt_c}\frac{\partial}{\partial \bar{s}}\left(\frac{G}{L}\right) + \frac{\theta^2}{Lt_c}\frac{\partial}{\partial \bar{s}}\left\{\frac{1}{Lt_c}\frac{\partial}{\partial \bar{s}}\left(\frac{G}{L}\right)\right\} + \dots, \tag{2.6}$$

where the derivatives are all evaluated at the point $\bar{s}$. Equation (2.6) is still very complicated as it involves f on both sides. If we now further expand in θ, writing

$$f = f^{(0)} + \theta f^{(1)} + \theta^2 f^{(2)} + \dots, \tag{2.7}$$

$$L(f) = L(f^0) + \theta L(f^1) + O(\theta^2), \tag{2.8a}$$

$$G(f) = G(f^{(0)},f^{(0)}) + \theta[G(f^{(0)},f^{(1)}) + G(f^{(1)},f^{(0)})] + O(\theta^2), \tag{2.8b}$$

substitute these expansions into (2.6) and collect terms of equal order in θ, we get first

$$f^{(0)} = G(f^{(0)}, f^{(0)})/L(f^{(0)}), \tag{2.9}$$

which means that the collision terms for $f^{(0)}$ are zero and hence that $f^{(0)}$ is the Maxwellian

$$f^{(0)} = n(\beta/\pi)^{\frac{3}{2}} \exp\{-\beta(\mathbf{v}-\mathbf{u})^2\} \equiv F. \tag{2.10}$$

(We use n for number density and $\mathbf{u}$ for gas velocity; $\beta^{-\frac{1}{2}}$ is the most probable thermal speed.)

Similarly, we find that the next higher order term $f^{(1)}$ is governed by the equation

$$G(F, f^{(1)}) + G(f^{(1)}, F) - FL(f^{(1)}) - f^{(1)}L(F) = -\frac{1}{t_c}\frac{\partial}{\partial \bar{s}}F, \tag{2.11}$$

where we have used (2.10). Equation (2.11) is related to the integral equation obtained for the so-called 'second approximation' in the Chapman–Enskog procedure, which we shall briefly describe for a model Boltzmann equation in the next section. The chief difference is that in the Chapman–Enskog procedure, the differential operator on the right of (2.11) is also expanded in θ; keeping the lowest term in this expansion amounts to assuming that the flow is described to a first approximation by Euler's (inviscid) equations, and leads to the Chapman–Enskog results.

The present approach brings out the following points. In the first place (2.11) is here obtained as a step in an asymptotic calculation, thus showing explicitly the asymptotic nature of the Chapman–Enskog expansion. More importantly, it is clear that whether the asymptotic evaluation of the integral in (2.3) is useful will in general depend on the particular characteristic. Now σ is proportional to the time of flight at velocity $\mathbf{v}$ from an initial point (where, for example, f may have been given) to the 'field' point $(\mathbf{x}, t)$ (where f is desired). As $\mathbf{v}$ takes all possible values to infinity, it is possible that there will always be sufficiently high values of v, at any given $(\mathbf{x}, t)$, such that σ would be small; and hence such that the first term in (2.3) would not be negligible and that the second would not have a valid asymptotic development of the form (2.6). In fact, for small σ, (2.3) indicates that f will in general be given by integrals over s, rather than in terms of local derivatives as in (2.6). The distribution at small σ thus corresponds to a kind of free molecule flow; and this is so, irrespective of how slow the flow variations are, i.e. how small θ is. In other words, the solution of (2.11) cannot be uniformly valid in velocity space; it provides only an 'inner' asymptotic expansion (as we shall call it) for the distribution.

This conclusion is confirmed by an examination of the solution itself, as we shall see in the next section; and it has a simple physical explanation. For molecules which are centres of force, the scattering angles are on an average smaller at higher velocities. Very fast molecules thus do not suffer many effective collisions, and have larger free paths. One can, if one wishes, define a velocity-dependent Knudsen number, which is arbitrarily high for sufficiently high velocities, no matter what the 'mean' (i.e. the conventional) Knudsen number is. Fast molecules therefore tend to remain in a sort of free-molecule flow.

We shall now illustrate these general conclusions by the detailed discussion of the flow in a weak shock, using a simple collision model.

3. The classical solution for the weak shock

For future reference, and to bring out certain points more explicitly, we give a brief résumé of the classical theory of the weak shock. We assume a simple relaxation model for the collision terms (Bhatnagar, Gross & Krook 1954; Welander 1954) as this simplifies the mathematics to some extent, so that we can write the model Boltzmann equation as

$$\left(\frac{\partial}{\partial t}+\mathbf{v}.\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x},t;\mathbf{v}) = \alpha(F-f). \tag{3.1}$$

Here α is supposed known as a function of the state of the gas, but for a weak shock it is adequate to treat it as some given constant. A more detailed discussion of the significance of the model in general and of the value of α can be found in Liepmann, Narasimha & Chahine (1962, cited as LNC below), and Narasimha (1961).

It is convenient to write down here what we shall call the equations of motion, which are obtained by multiplying (2.1) or (3.1) with the collisional invariants 1, v and v^2 and integrating over all velocities:

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho u_i) &= 0,\\ \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right) &= -\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j},\\ \frac{\partial \beta}{\partial t}+u_i\frac{\partial \beta}{\partial x_i} &= \frac{4\beta^2}{3\rho}\left[\frac{\partial q_i}{\partial x_i}-\tau_{ij}\frac{\partial u_i}{\partial x_j}+p\frac{\partial u_i}{\partial x_i}\right],\\ p &= \rho/2\beta. \end{aligned}\right\} \tag{3.2}$$

Here ρ is the density, p the pressure, τ_{ij} the additional stress, and q_i the heat flow vector; and the summation convention has been followed in the notation.

The classical Chapman–Enskog procedure for solving (3.1) in a problem like the structure of a weak shock may be described as follows. One first extracts from (3.1), by obvious non-dimensionalization, a small parameter

$$\theta = t_c/t_f = (\alpha t_f)^{-1}$$

where t_c, the collision time characteristic of the right-hand side of (3.1), can be taken to be α^{-1}. It is then assumed, as in the previous section, that f can be expanded as the series (2.7) in θ; this series is now substituted in (3.1), and collecting terms of equal order in θ and using the equations of motion as described in detail by Chapman & Cowling (1952) for the full Boltzmann equation, one obtains

$$f^{(0)} = F, \tag{3.3}$$

$$f^{(1)} = \frac{F}{\alpha}\left[\frac{c_k}{\beta}\frac{\partial \beta}{\partial x_k}(\beta c^2-\tfrac{5}{2})-2\beta\frac{\partial u_k}{\partial x_l}(c_k c_l-\tfrac{1}{3}c^2\delta_{kl})\right], \tag{3.4}$$

etc. where $c_k = v_k-u_k$ is the peculiar velocity. Thus the zeroth order term corresponds to local equilibrium; and the first order term is a correction for the departure from equilibrium in terms of velocity and temperature gradients.

From the solution (3.4), one can easily show that the stresses and heat flux are related linearly to the velocity and temperature gradients through the usual transport parameters; for the model under study, we obtain a kinematic viscosity $\nu = \frac{1}{2}\alpha\beta$. Use of these parameters in (3.2) yields the Navier–Stokes equations for a monatomic Stokesian gas with Prandtl number unity.

The Navier–Stokes equations were solved for steady flow through a plane weak shock by Taylor (1910). If we define

$$N \equiv (n-n_1)/(n_2-n_1), \quad B \equiv (\beta-\beta_1)/(\beta_2-\beta_1), \tag{3.5}$$

subscripts 1 and 2 denoting conditions at upstream and downstream infinity (see figure 1), we can write Taylor's solution as

$$N_T(\xi) = B_T(\xi) = \tfrac{1}{2}(1+\tanh\tfrac{1}{2}\xi), \tag{3.6a}$$

$$\xi \equiv \tfrac{2}{3}\epsilon u_1 x/\nu = \tfrac{4}{3}\epsilon\alpha\beta u_1 x, \quad \epsilon \equiv (n_2-n_1)/n_1, \tag{3.6b}$$

using values appropriate to the model. For a weak shock, $u_1 \simeq (5RT/3)^{\frac{1}{2}}$, the speed of sound in a monatomic gas.

The shock thickness as given by (3.6) is $O(\epsilon^{-1})$, and hence the gradients within the shock are $O(\epsilon^2)$. This is consistent with the use of the Chapman–Enskog expansion, (3.3) and (3.4). There is, in fact, little doubt that the gross features of the structure of a weak shock are described adequately by the Navier–Stokes theory, and both theory and experiment lend support to this view. Thus, the computations made on the full non-linear model Boltzmann equation (reported in LNC) showed that for weak shocks (Mach number $M \lesssim 2{\cdot}0$), the structure was essentially Navier–Stokes; the work of Darrozès (1963) supports this conclusion. Also the experiments of Sherman (1955) have shown that the profile and the thickness of the shock agree well with Navier–Stokes results at $M \lesssim 2{\cdot}0$.

However, there are certain difficulties with the Chapman–Enskog procedure. First, the expansion (2.7) implies that the perturbation on $f^{(0)}$ is small for all velocities $\mathbf{v}$, i.e. that $\theta f^{(1)} \ll f^{(0)}$ uniformly in velocity space. From the solution (3.4), however, this is immediately seen to be not so; for any given departure from equilibrium (a measure of which is presumably either of the gradients $\partial\beta/\partial x_l$ or $\partial u_k/\partial x_l$), however small, one can always find large enough values of the velocity c_k such that the perturbation is not small, and even such that the total distribution is negative (see for example, figure 2). If the departure from equilibrium is large, as within a strong shock (see, for example, LNC), this difficulty with the Chapman–Enskog solution can become severe. For example, Sherman & Talbot (1960) show such a calculated distribution at a Mach number of 5·2, with f going negative at quite small values of the velocity. However, the velocities at which the perturbation becomes appreciable grow larger as the departure becomes smaller; hence moments of f, of a given order, may not be affected seriously if f is already small at the high velocities. But this has to be demonstrated.

Secondly, while solving for any term in the series (2.7), Chapman and Enskog use the equations of motion corresponding to the previous approximation to f. Thus, in computing $f^{(1)}$, the time derivatives of the flow quantities (required for evaluating $\partial F/\partial \bar{s}$ in (2.11)) are supposed given to some approximation by the equations of motion corresponding to $f = f^{(0)} = F$, i.e. the inviscid Euler equations

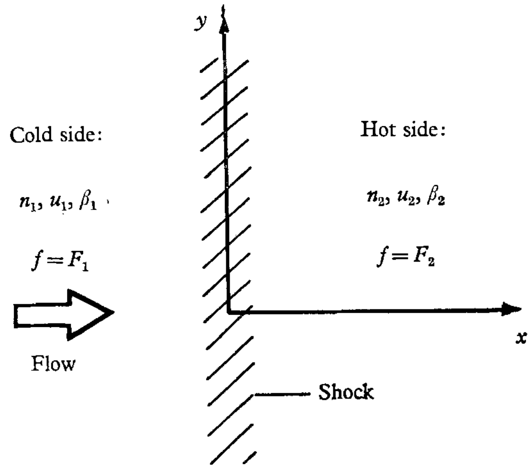

FIGURE 1. Co-ordinate system and notation.

obtained by putting $q_i = \tau_{ij} = 0$ in (3.2) (Chapman & Cowling 1952, p. 112). This implies that the Euler equations are in some sense a first approximation to any flow, and that the Navier–Stokes equations, which correspond to $f = F + \theta f^{(1)}$, are only a small correction to them. But it is well known that there are many gas flows accounted for very well by the Navier–Stokes equations but not even qualitatively by the Euler equations. It therefore seems that the Chapman–Enskog procedure is perhaps too restrictive, although not inconsistent.

A closer study of the distribution function is clearly necessary to bring out the consequences of these limitations of the Chapman–Enskog theory.

4. Formulation of basic integral equation

For steady one-dimensional flow through a plane shock layer, (3.1) reduces to

$$v_x \frac{\partial f}{\partial x} = \alpha(F - f), \tag{4.1}$$

with flow taken to be along the x-axis in a shock-fixed co-ordinate system. There are now only two families of characteristics, corresponding to $v_x \gtrless 0$; thus the integral equation (2.3) becomes, using the model collision terms,

$$f_{\pm} \equiv f(x, v_x \gtrless 0) = \int_{\mp\infty}^{x} \frac{\alpha F(x; \mathbf{v})}{v_x} \exp\left[-\frac{\alpha(x-x')}{v_x}\right] dx'. \tag{4.2}$$

The first term on the right of (2.3) does not appear in (4.2), as the boundary condition is at infinity and does not contribute to the solution.

At $v_x = 0$, we have of course $f_+ = f_- = F$, from (4.1). Thus the very slow molecules are in equilibrium in a sense, and this is clearly true for the general model equation (3.1) in any flow, and is not particular to the weak shock.

Before proceeding with further analysis, it is useful to simplify and rewrite the basic integral equation (4.2). First we note that we can integrate out the transverse components v_y, v_z of the velocity vector from (4.2), and work only in terms of the contracted distributions

$$f(x, v_x) \equiv \int f(x, \mathbf{v})\, dv_y\, dv_z, \tag{4.3a}$$

$$F(x, v_x) \equiv \int F(x, \mathbf{v})\, dv_y\, dv_z = n(\beta/\pi)^{\frac{1}{2}} \exp[-\beta(v_x - u)^2]. \tag{4.3b}$$

We shall use the same symbols f, F also for the contracted distributions, as the presence of only a single component v_x in their argument is enough to differentiate them from the full distributions. Secondly, we introduce the non-dimensional variables

$$V = v_x\beta_1^{\frac{1}{2}}, \quad U = u\beta_1^{\frac{1}{2}}, \quad C = V - U, \quad \eta = (\tfrac{4}{3}\epsilon U_1 V)^{-1}, \tag{4.4}$$

where, for a weak shock, $U_1^2 \simeq \frac{5}{6}$ in a monatomic gas. Further, the quantities n, β and u in (4.3) can be written in terms of the normalized variables N and B of (3.5), using the Rankine–Hugoniot relations for a weak shock (i.e. $\epsilon \to 0$). Finally we change from x to the non-dimensional variable ξ of (3.6*b*). In these new variables (4.2) takes the form

$$f_\pm = \eta \int_{\mp\infty}^{\xi} F(\xi', V; \epsilon) \exp\{-\eta(\xi - \xi')\} d\xi' \tag{4.5}$$

where
$$F(\xi', V; \epsilon) = n_1(\beta_1/\pi)^{\frac{1}{2}}(1 + \epsilon N)\{1 + (1 - \tfrac{2}{3}\epsilon + \tfrac{13}{9}\epsilon^2)B\}^{\frac{1}{2}}$$
$$\times \exp\{-(1 - \tfrac{2}{3}\epsilon + \tfrac{13}{9}\epsilon^2) BC^2 + O(\epsilon^3)\} + O(\epsilon^3), \tag{4.6}$$

and we have retained terms up to $O(\epsilon^3)$ in evaluating the jump relations across the shock. It may be emphasized that although the choice of ξ as the independent variable is inspired by the Navier–Stokes solution, this solution itself is not used anywhere in writing (4.5) and (4.6). For F in (4.5), we shall use either of the representations (4.3*b*), (4.6) as proves convenient.

5. Inner solution

We now study different limits of the solution of (4.5), in the spirit of the perturbation techniques developed by Kaplun, Lagerstrom and Cole (see, for example, Van Dyke 1964). First we define an inner limit as the process $\epsilon \to 0$, V fixed (or equivalently $C = V - U$ fixed, as $U = O(1)$). The exponent in (4.5) contains in this limit a large number $\eta = O(\epsilon^{-1})$, and using Laplace's method we obtain an asymptotic expansion for the inner solution, in inverse powers of ϵ:

$$f_\pm = F(\xi) - \frac{1}{\eta}\frac{\partial F}{\partial \xi} + \dots + (-\eta)^{-n}\frac{\partial^n F}{\partial \xi^n} + O(\eta^{-n-1}). \tag{5.1}$$

This will be recognized as a specialization of the expansion (2.6) to a particular model and flow. To see its relation to the Chapman–Enskog solution, we may use the representation (4.3*b*) for F and write

$$\frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial n}\frac{dn}{d\xi} + \frac{\partial F}{\partial u}\frac{du}{d\xi} + \frac{\partial F}{\partial \beta}\frac{d\beta}{d\xi}.$$

Employing the steady one-dimensional version of the complete equations of motion (3.2), we can eliminate n and u from $\partial F/\partial \xi$, and write the second term in the expansion (5.1) as

$$\frac{1}{\eta}\frac{\partial F}{\partial \xi} = \tfrac{4}{3}\epsilon U_1 \beta_1^{\frac{1}{2}} F \left[\frac{2}{3}\frac{du}{d\xi}(2\beta c^2 - 1) + \frac{c}{\beta}\frac{d\beta}{d\xi}(\tfrac{3}{2} - \beta c^2) + \frac{2}{3}\frac{\beta}{\rho}\left\{c\frac{d\tau}{d\xi} + (1 - 2\beta c^2)\left(\frac{dq}{d\xi} - \tau\frac{du}{d\xi}\right)\right\}\right], \tag{5.2}$$

where we have omitted subscripts on τ and q as only one component of these quantities is relevant in our one-dimensional problem. It is interesting to note that the continuity equation alone is enough to write the second term of (5.1) in terms of the two gradients $du/d\xi$, $d\beta/d\xi$; but the expression so obtained contains u explicitly, and so would not be Galilean-invariant with respect to the gas velocity, as (5.2) is.

The first two terms within the brackets on the right of (5.2) are $O(\epsilon)$; if τ/p and $\beta^{\frac{1}{2}}\Delta q/p$ are $o(\epsilon)$, then the last two terms, enclosed within curly brackets, can be ignored compared with the first two, which can be recognized as the contracted form of the Chapman–Enskog terms (3.4). If we compute τ and q from the first two terms, as in § 3, then we find that the last two terms are indeed $O(\epsilon^2)$ and so negligible in the limit. Hence for a weak shock the inner and the Chapman–Enskog solutions are equivalent to the lowest order.

It is convenient for later work to rewrite the lowest order terms in the inner solution in the non-dimensional variables (4.4). Noting that $\beta c^2 \simeq \beta_1 c^2 = C^2$, we will have, from (5.1) and (5.2),

$$f_{\pm} = F[1 + \tfrac{20}{27}\epsilon^2\{N'(2C^2 - 1) + (B'/U_1)C(\tfrac{3}{2} - C^2)\} + O(\epsilon^3)], \tag{5.3}$$

where the dashes on N and B denote derivatives with respect to ξ. Equation (5.3) can also be obtained directly from (5.1) using the expression (4.6) for F.

The term in N' in (5.3) is responsible for the stress τ, and the term in B' for the heat flux q. It is evident that this solution is not valid for arbitrarily large C; the heat transfer term becomes $O(1)$ for $C = O(\epsilon^{-\frac{2}{3}})$. For C greater† than this, f_+ as given by (5.3) will even become negative, indicating the need for a different asymptotic expansion.

If we calculate a moment of the distribution using (5.3), the contribution from velocities greater than $\epsilon^{-\frac{2}{3}}$ to a moment of order n (i.e. to the integral $\int f V^n dV$) is easily shown to be $O(\epsilon^{-(n-10)/6}\exp\{-\epsilon^{-\frac{1}{3}}\})$. Hence for a moment of given order, the use of the inner solution over regions of velocity space where it is not valid

† It is convenient to interpret the words 'greater than' and 'less than' in an order-of-magnitude sense. Thus C greater than $\epsilon^{-\frac{2}{3}}$ implies that we consider the limit $\epsilon^2 C^3 \to \infty$.

yields contributions which are exponentially small in ϵ. But for fixed ϵ, these contributions increase with the order of the moment and are not negligible when

$$(n-10)/6 \sim (\epsilon^{\frac{1}{3}} \ln \epsilon)^{-1}.$$

We can make some general remarks at this stage on the inner solution. First, it is clear that even in a strong shock, for which ϵ is not small, the asymptotic argument which led to (5.1) and (5.2) is valid for sufficiently small velocities, because the exponent in (4.5) is inversely proportional to the velocity. Therefore, (5.2) always gives the correction to F at sufficiently low velocities. Hence, although for a weak shock the correction remains small, and so the solution (4.9) remains valid, even for C quite large (it has only to be less than $\epsilon^{-\frac{2}{3}}$), in the more general context it is convenient to think of the inner solution as the limiting solution for relatively low velocities. However, for a strong shock τ/p and $q\beta^{\frac{1}{2}}/p$ are not small (as shown by LNC), and hence the last two terms in (5.2) cannot be ignored. So, while the inner solution would still be known in terms of the local flow, the problem cannot be closed and in particular does not lead to the Navier–Stokes equations. A second reason why the Navier–Stokes equations would not be valid is of course that even the complete inner solution (5.2) would cover only a small part of velocity space, and so cannot yield the correct moments.

Secondly, in deducing (5.3) we have not appealed to the Euler equations as a first approximation, as the Chapman–Enskog procedure does. It was only assumed, and later verified, that τ/p and $q\beta^{\frac{1}{2}}/p$ were small. In particular, nothing was said about the relative magnitude of the different terms which appear in the momentum equation—in fact (see, for example, Lighthill 1956) they are all of the same order within the shock! Thus, the Chapman–Enskog decomposition of the time derivative in (2.11) has been avoided.

Finally (5.1) shows that our inner solution, and the distribution function at low velocities in general, depend only on the local flow, i.e. on the flow quantities and their derivatives. This gives the inner solution a Markovian character; and when the inner solution is a sufficiently good approximation to the complete solution, it leads to the Navier–Stokes equations, in which the diffusive terms (containing viscosity and heat conductivity) are a direct consequence of the Markov properties of the inner solution.

6. Distribution of fast molecules moving downstream

To obtain the distribution at arbitrarily high velocities, we have to make a distinction between molecules moving downstream and those moving upstream. This is to be expected from the considerations of § 2, where it was shown that at sufficiently high velocities the distribution is, in general, determined by integrals over the flow.

We will actually find it necessary to construct two further limits below to obtain a complete description of the distribution. It is again convenient to think of these two limits together as corresponding to fast molecules, interpreting the word 'fast', in a relative sense, to mean those velocities for which the inner solution is not valid.

6.1. *The outer limit*

We define this as the process $\epsilon \to 0$, η fixed (or equivalently, ϵV fixed). Extracting a factor F_1 from the right side of (4.6), and substituting for C in terms of η from (4.4), the outer approximation to F can be written, after some algebra, as

$$\frac{F}{F_1} = \exp\left\{\frac{9}{20}\frac{B}{\epsilon\eta^2} - \frac{B+\frac{3}{2}N}{\eta}\right\} + O(\epsilon). \tag{6.1}$$

Thus, with $B > 0$, F/F_1 is exponentially great in the outer limit (although F itself is small). Physically, this means that because of the increase in temperature as we move downstream across the shock, the number of particles scattered into the high velocities at a given point is much larger than at any point upstream of it.

Putting (6.1) into (4.5), we obtain the outer integral equation to $O(\epsilon)$,

$$f_+ = \eta F_1 \int_{-\infty}^{\xi} d\xi' \exp\left\{\frac{9B(\xi')}{20\epsilon\eta^2} - \frac{B(\xi')+\frac{3}{2}N(\xi')}{\eta} - \eta(\xi-\xi')\right\}. \tag{6.2}$$

If B and N are $o(\epsilon)$, we get $f_+ \approx F_1$ to the lowest order, and local equilibrium is then a good approximation also in the outer limit.

Consider now points within the shock, with $B = O(1)$. An asymptotic estimate of the integral (6.2) yields the lowest order outer solution as

$$f_+ \approx \frac{9}{20(dB/d\xi)}\epsilon\eta^3 F. \tag{6.3}$$

Thus at very high velocities ($\eta \to 0$) the distribution falls faster than the local Maxwellian by the inverse cube of the velocity; and the perturbation from the local Maxwellian, say $(f_+ - F)/F$, tends to the value -1.

It is worth considering briefly the physical significance of this simple result. In obtaining it, we have found that the second and third terms in the exponential in (6.2) are negligible. This means, in particular, that the attenuation due to collisions (represented by the third term) is negligible in the outer limit, which we may therefore class as collisionless. But in contrast to normal collisionless flow, the outer solution does not represent a memory of an upstream distribution (which would be F_1 in the present problem). This peculiarity is due to the relatively much more intense 'creation' (by collisions!) of fast particles at the hotter places, which feature, as we have already noted, is a direct consequence of (5.2). And once created, these fast particles are hardly ever lost by collisions as their free paths are large, and so they accumulate by streaming. This result may seem at first sight to be due to the particular collision model we have chosen; but clearly it provides a specific description of a much more general phenomenon which is an essential part of the irreversible process that occurs within a shock: namely, the transfer of molecules from low velocities to high velocities with increase in temperature as we move downstream across the shock. This conclusion is supported by the more detailed comparison with other models which we shall undertake in §8.

A further consequence of this mechanism is that the outer solution for f_+ turns out in this particular case to be local, like the inner solution. We shall see in §7 that this is not true for f_-.

We see, therefore, that even in a weak shock, in which the departure from equilibrium is nominally small, the distribution at a given point contains features characteristic of both continuum and collisionless flows (even if the collisionless flow is rather peculiar). We found previously that the inner solution (describing what one may call the continuum core) leads to large negative perturbations at high velocities. The outer solution (6.3) similarly leads to infinities at low velocities ($\eta \to \infty$). A simple matching of these solutions is not therefore possible, and an intermediate limit covering the transition region is necessary.

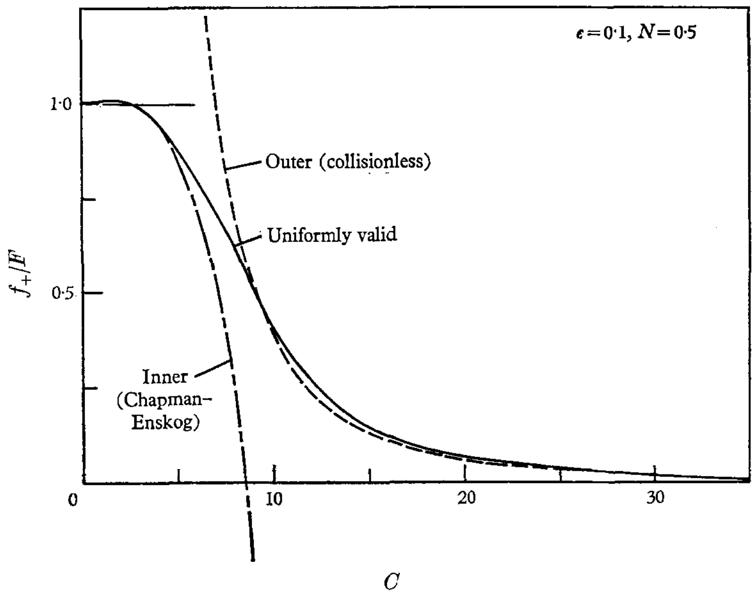

FIGURE 2. The distribution function at the centre of the shock for molecules moving downstream. The inner or Chapman–Enskog curve is from (5.3) and is valid at relatively low velocities. The collisionless solution is from (6.3) and is valid only at extremely high velocities. The uniformly valid solution, (6.6), contains both these and the intermediate one.

6.2. *The intermediate limit*

This limit process can be defined as $\epsilon \to 0$, $\zeta \equiv \epsilon^{\frac{1}{3}}\eta$ fixed. Carrying it out on F in (4.6), we can write the intermediate limit of the integral equation as

$$f_+ = \epsilon^{-\frac{1}{3}}\zeta F \int_{-\infty}^{\xi} d\xi' \exp\left\{-\epsilon^{-\frac{1}{3}}\left[\frac{9}{20}\frac{B - B(\xi')}{\zeta^2} + \zeta(\xi - \xi') + O(\epsilon^{\frac{2}{3}})\right]\right\}. \tag{6.4}$$

Estimation of this integral by standard methods gives the intermediate solution

$$f_+ \approx F\left(1 + \frac{9}{20\zeta^3}\frac{dB}{d\xi}\right)^{-1}, \tag{6.5}$$

which again is local. The outer limit of this solution ($\zeta \to 0$) agrees with the outer solution (6.3) (which is in fact contained in the intermediate solution). Further-more, the inner limit of (6.5), obtained by letting $\zeta \to \infty$, is in just the right form to match the intermediate limit of the inner solution (5.3). It is now possible to construct a uniformly valid solution; e.g. we can write

$$f_+ \approx F\left(1+\frac{20}{27}\frac{dB}{d\xi}\epsilon^2 C^3\right)^{-1}\left[1+\tfrac{20}{27}\epsilon^2\left(1+\frac{20}{27}\frac{dB}{d\xi}\epsilon^2 C^3\right)^{-1}\left\{\frac{dN}{d\xi}(2C^2-1)+\frac{3}{2}\frac{C}{U}\frac{dB}{d\xi}\right\}\right]. \tag{6.6}$$

It is easily verified that this expression reduces to the lowest order inner, intermediate and outer solutions in the respective limits. It is possible to write a uniformly valid solution in many different ways, but all these will be essentially equivalent to (6.6) to the proper order. Figure 2 shows the inner, outer and uniformly valid solutions at a typical point within the shock.

7. Distribution of fast molecules moving upstream

To consider $f_- = f(V < 0)$, it is convenient to transform the variables of §6 to new ones defined by

$$\eta_- \equiv -\eta, \quad \zeta_- \equiv -\zeta, \quad N_- \equiv 1-N, \quad B_- \equiv 1-B, \tag{7.1}$$

so that we work with positive quantities; as our characteristic is now directed left from $\xi = +\infty$, N_- and B_- increase along it, from zero at $\xi = +\infty$ to 1 at $\xi = -\infty$.

7.1. *The outer limit*

This is defined as before, with $\epsilon \to 0$ as η_- is fixed. The outer approximation to F is

$$\frac{F}{F_2} = \exp\left[-\frac{9}{20}\frac{B_-}{\epsilon\eta_-^2}-\frac{B_-+\frac{3}{2}N_-}{\eta_-}\right][1+O(\epsilon)], \tag{7.2}$$

and the outer integral equation is

$$f_- = \eta_- F_2\int_\xi^\infty d\xi' \exp\left[-\frac{9B_-(\xi')}{20\epsilon\eta_-^2}-\frac{B_-(\xi')+\frac{3}{2}N_-(\xi')}{\eta_-}-\eta_-(\xi'-\xi)+O(\epsilon)\right]. \tag{7.3}$$

The most significant difference between this and the corresponding equation (6.2) for f_+ is that the exponent in the integrand of (7.3) is negative, reflecting the fact that the most intense scattering into high velocities occurs at downstream infinity, where the temperature is highest. This suggests that the greatest contribution to the integral (7.3) comes from $B_-(\xi') \to 0$ but in this limit $\xi' \to +\infty$ and hence the attentuation (the last term in the exponent) would also be strong. To evaluate this effect, we first rewrite (7.3), retaining only the strongest terms, as

$$\left.\begin{aligned} f_- &\approx F_2\eta_- Z^{-\eta_-}\int_0^Z \exp\left[-\frac{9}{20}\frac{B_-}{\epsilon\eta_-^2}\right]Z'^{\eta_- -1}\,dZ', \\ Z &\equiv e^{-\xi}. \end{aligned}\right\} \tag{7.4}$$

To estimate this we need to know the asymptotic form of $B_-(Z')$ as $Z' \to 0$. This is obtained by the following argument. First, from the previous section, we have $f_+ < F$ in both the intermediate and outer limits. From (7.3), we see that $f_- < F_2$

in these limits (it is clear that as for f_+ the intermediate limit for f_- will again balance the first and third terms in (7.3); we shall confirm this in §7.2). Thus, all limits tend to the local Maxwellian as $\xi \to +\infty$. However, while the contributions to the moments are algebraic in ϵ from the inner solution, it is easily shown that they are exponentially small from the other two solutions, as they are valid only for velocities greater than $\epsilon^{-\frac{2}{3}}$. Secondly, we have already seen that the use of the inner solution over the part of velocity space where it is not valid leads only to exponentially small contributions in moments of a given order. The result of these considerations is that as $\xi \to +\infty$ the inner and consequently the Navier–Stokes solutions yield correctly the lowest order contributions to the moments. (It must be emphasized that this argument is not valid for $\xi \to -\infty$, because as we shall see below in this limit the outer solution does not tend to the local Maxwellian.)

Thus $B \simeq B_T$, and from (3.6*a*) it now follows that $B_-(Z') \approx Z'$ as $Z' \to 0$. Hence (7.4) is asymptotically

$$f_- \approx F_2 \eta_- (20\epsilon\eta_-^2/9Z)^{\eta_-} \Gamma(\eta_-, 9Z/20\epsilon\eta_-^2), \tag{7.5}$$

where

$$\Gamma(\alpha, x) \equiv \int_0^x e^{-t} t^{\alpha-1} dt. \tag{7.5a}$$

For $Z = o(\epsilon)$, (7.5) gives $f_- \approx F_2$ as may be expected. For $Z = O(1)$, it reduces to

$$f_- \approx F_2 e^{\xi\eta_-} (20\epsilon\eta_-^2/9)^{\eta_-} \eta_- \Gamma(\eta_-), \tag{7.6}$$

where we have returned to ξ as the independent variable and $\Gamma(\alpha)$ denotes the complete gamma function $\Gamma(\alpha, \infty)$ in (7.5*a*).

The simple result (7.6) is again very revealing. It shows that, for given ξ and η_-, $f_-/F_2 = O(\epsilon^{\eta_-})$; since F_2/F is exponentially great in ϵ, this means that the perturbation on the *local* Maxwellian, say $(f_- - F)/F$, is also exponentially great in the outer limit! Further, taking the limit of (7.6) as $\eta_- \to 0$,

$$f_- \approx F_2[1 - \gamma\eta_- + 2\eta_- \ln\{(20\epsilon/9)^{\frac{1}{2}}\eta_-\} + O(\eta_-^2 \ln \eta_-)] = F_2 \quad \text{as} \quad \eta_- \ln \epsilon \to 0, \tag{7.7}$$

where $\gamma = 0{\cdot}577\ldots$ is the Euler–Mascheroni constant. Thus for sufficiently high velocities (from (4.4) the limit in (7.7) requires more specifically that V be greater than $\epsilon^{-1} \ln \epsilon$), the distribution corresponds to the downstream Maxwellian! The departure from F_2 for small but non-zero η_- is non-analytic and contains logarithmic terms.

Comparing this result with (6.3), we see that for molecules moving upstream the memory of the downstream Maxwellian F_2 is dominant at sufficiently high velocities, but it is not in general collisionless for $\eta_- = O(1)$. The nature of the solution is hence global, in contrast to the outer solution for f_+.

7.2. *The intermediate limit*

In this limit ζ_- is fixed as $\epsilon \to 0$, and the relevant integral equation is

$$f_- = \epsilon^{-\frac{1}{3}} \zeta_- F_2 \int_\xi^\infty d\xi' \exp\left\{-\epsilon^{-\frac{1}{3}} \left[\frac{9B_-(\xi')}{20\zeta_-^2} + \zeta_-(\xi' - \xi)\right]\right\}. \tag{7.8}$$

This integral can be evaluated asymptotically by the saddle-point method; as the location of the saddle point (say ξ_0) is a function of ζ_-, the discussion becomes rather involved, so we content ourselves with quoting some of the main results. The general features of the solution can in fact be guessed from (7.8), as we know that B_- decreases monotonically to zero as $\xi \to +\infty$. It is found that as ζ_- increases from zero, the saddle point moves in from $+\infty$, till at a certain critical value of ζ_-, say ζ_-^*, it coincides with ξ. For $\zeta_- > \zeta_-^*$, the largest contribution to the integral comes from a neighbourhood of ξ itself. The solutions take the following form. For $\zeta_- < \zeta_-^*$, $\xi_0 > \xi$,

$$\left.\begin{aligned} f_- &\approx \left(\frac{40}{9B_0''}\epsilon^{-\frac{1}{3}}\right)^{\frac{1}{2}} \zeta_-^2 F(\xi_0) \exp\{-\epsilon^{-\frac{1}{3}}\zeta_-(\xi_0-\xi)\}, \\ B_0'' &\equiv (d^2B/d\xi^2)_{\xi=\xi_0}. \end{aligned}\right\} \tag{7.9}$$

This represents an attenuated sample of the Maxwellian at ξ_0, so to speak. The outer limit of (7.9), obtained by letting $\zeta_- \to 0$ and using the Navier–Stokes result for B_0'' as $\xi_0 \to +\infty$ for reasons set forth earlier, is found to be

$$f_- = (2\pi\eta_-)^{\frac{1}{2}} F_2 \exp[-\eta_-\{1-\xi-\ln(20\epsilon\eta_-^3/9)\}] \tag{7.10}$$

in outer variables; and this can be shown to match the intermediate limit of the outer solution (7.6), making use of Stirling's formula for the asymptotic of the gamma function. Thus, the point at which F is sampled by the intermediate solution is at downstream infinity for extremely high velocities, and moves closer at lower velocities.

For velocities below the critical value, $\zeta_- > \zeta_-^*$, (7.8) can be evaluated by Watson's lemma, and yields

$$f_- \approx F\left(1+\frac{9}{20\zeta_-^3}\frac{dB}{d\xi}\right)^{-1}, \tag{7.11}$$

which is identical with the intermediate solution (6.5) for f_+. It follows that it can be matched with the intermediate limit of the inner solution, as in § 6.

The possibility of matching the different solutions shows that for both f_+ and f_- a consideration of the three limits we have proposed covers the whole of velocity space. We are now in a position to evaluate the moments of f at any point. However, by the same kind of argument as we used in § 7·1 to estimate the asymptotic form of $B(Z')$, it is easily shown that, for ξ fixed, $\epsilon \to 0$, the dominant contribution to the moments comes from the inner solution. That is, in this limit, the lowest order approximation to the solution for moments of a given order is the same as the Navier–Stokes approximation. However, this solution is not necessarily uniformly valid in ξ as we shall see in the next section.

7.3. *The 'precursor'*

It is now interesting to consider the limit $\xi \to -\infty$. In a co-ordinate system in which a shock advances into fluid at rest, this limit corresponds to the region far ahead of the shock.

The inner solution (5.3) shows a perturbation on the local Maxwellian of $O(\epsilon^2 N')$, where N' is exponentially small in ξ. Also, the local Maxwellian is itself a perturbation of $O(\epsilon N)$ on F_1. Thus

$$f_\pm = F_1[1+O(\epsilon N)][1+O(\epsilon^2 N')]. \tag{7.12}$$

The outer solution for f_+ is bounded above by $F = F_1[1+O(\epsilon N)]$. The outer solution for f_- has the behaviour, from (7.6),

$$f_-/F_2 = O[\epsilon^{\eta_-}\exp\{-\eta_-|\xi|\}].$$

This is again exponentially small in ξ for $\eta_- = O(1)$, except for very small η_-. Thus, (7.6) suggests consideration of the limit $\xi\eta_-$ fixed, $\xi/\ln\epsilon \to \infty$; this gives

$$f_-/F_2 = \exp\{-\eta_-|\xi|\}, \tag{7.13}$$

which of course represents just the attenuated downstream Maxwellian. The intermediate solutions will match at one end with (7.12), and at the other with (7.13). In between they yield attenuations of the local Maxwellians at all points within the shock, all of which are bounded above by F_2. This suggests that a rough approximation to the whole distribution function as $\xi \to -\infty$ is simply

$$\left.\begin{aligned} f &= F_1 + \mathscr{H}(-V)F_2\exp\{-\eta_-|\xi|\}, \\ &= F_1 + \mathscr{H}(-V)F_2\exp\{-|\xi^*/V|\}, \\ \xi^* &\equiv 3|\xi|/4\epsilon U_1, \end{aligned}\right\} \tag{7.14}$$

where $\mathscr{H}(x)$ is Heaviside's step function, equal to unity for $x > 0$ and to zero for $x < 0$. This expression displays the essential features of the distribution far upstream; the slow molecules are in local equilibrium ($f \simeq F_1$), whereas the very fast ones coming from the hot side carry a memory of their distribution ($f \to F_2$ as $\eta_- \to 0$). The result represents what may be called the precursor of the shock (picturing now a shock advancing into still fluid). As we have noted earlier, by far the largest number of particles being scattered into high velocities come from the downstream side of the shock, where the temperature is highest. These are attentuated by collisions as in any molecular beam; but as the free path is proportional to the velocity, and sufficiently far ahead of the shock the fast particles will seem to be created effectively at $\xi = 0$, we obtain the exponential factor shown in (7.14). It is interesting to note that for these molecules there is a continual loss but hardly any gain (once they are created on the hot side of the shock); in contrast, for the fast molecules moving towards the hot side ($V > 0$), there is cumulated gain but hardly any loss!

If we define moments Q^ν by

$$Q^\nu \equiv \int f(v_x)\,v_x^\nu\,dv_x, \tag{7.15}$$

then moments of the precursor distribution (7.14) will differ from their value Q_1^ν at equilibrium by an amount

$$\int_{-\infty}^{0} F_2\exp\{-|\xi^*/V|\}\,v_x^\nu\,dv_x.$$

This quantity can be expressed in terms of the functions

$$g_\nu(\xi^*, U) \equiv \int_0^\infty V^\nu\,dV\exp\{-(V-U)^2-\xi^*/V\}, \tag{7.16}$$

which have been studied and tabulated by Chahine & Narasimha (1964). Using the asymptotic expansions of g_ν given by them for large ξ^*, we imediately have

$$\begin{aligned} Q^\nu - Q_1^\nu &= n_2(\pi\beta_2^\nu)^{-\frac{1}{2}} g_\nu(\xi^*, U_2) \\ &\approx n_2(3\beta_2^\nu)^{-\frac{1}{2}} (\tfrac{1}{2}\xi^*)^{\nu/3} \exp\{-3(\tfrac{1}{2}\xi^*)^{\frac{2}{3}}\}. \end{aligned} \quad (7.17)$$

The result that the moments decay eventually like $\exp\{-\xi^{*\frac{2}{3}}\}$ rather than like $\exp\{-\xi\}$ (omitting numerical constants multiplying ξ^*, ξ) has been derived previously by Lyubarskii (1961), using a very different method. He has pointed out that all continuum approximations predict an $\exp\{-|\xi|\}$ decay, and hence will never give the correct asymptotic law. From the present analysis, it is clear that the outer solution is responsible for the slower decay; and hence that the inclusion of more terms in the inner (continuum) expansion will not necessarily ensure improved results always. It should, however, be pointed out that the flow quantities $Q^\nu - Q_1^\nu$ will in fact show a decay like $\exp\{-|\xi|\}$ in a region where $|\xi| \gg 1$ and $N = O(\epsilon)$, and only for much greater ξ (as $\xi \ln \epsilon \to \infty$ and $N = o(\epsilon)$) will the decay become somewhat slower, eventually being like $\exp\{-\xi^{*\frac{2}{3}}\}$.

The method we have used for obtaining these results gives us some insight into the structure of the distribution, and further suggests a way of generalizing them. For, we can derive the result (7.14) very simply from the basic integral equation (4.5). If we are far upstream of the shock in terms of the shock thickness, we can to a first approximation replace F by

$$F \simeq F_1 + (F_2 - F_1)\mathscr{H}(\xi); \quad (7.18)$$

i.e. an observer far away sees very nearly a discontinuity at $\xi = 0$. Putting (7.18) into (4.5) and recalling that $F_2 \gg F_1$ at high velocities, we have

$$f \simeq F_1 + \mathscr{H}(-V) F_2 \exp\{-|\xi|\eta_-\}. \quad (7.19)$$

This is in agreement with (7.14), showing that replacing the gain term in (4.15) by a discontinuity leads to a useful asymptotic result for the distribution far upstream.

8. Extension to the true Boltzmann equation

In the model we have employed, the loss term has the same form as one would expect for Maxwell molecules in the true Boltzmann equation, and so is fairly realistic. But the assumption that the gain is proportional to the local Maxwellian rests on weaker ground, and the conclusions which can be traced to this assumption are precisely those on which doubts may be raised.

Thus, for the true equation one can still say that at sufficiently low velocities $G(f) - fL(f) \simeq 0$, but this is not enough to conclude that the distribution at low velocities is approximately Maxwellian, because $G(f)$ is a complicated integral over all $\mathbf{v}$. However, there is not much doubt that the very slow molecules are collision-dominated, because they always have very small free paths.

For the true Boltzmann equation our basic integral equation for the shock can be written as

$$f_\pm(x, v) = \int_{\mp\infty}^{x} dx' \frac{G(x', v)}{v_x} \exp\left\{-\int_{x'}^{x} L(x'', v)\frac{dx''}{v_x}\right\}, \quad (8.1)$$

in analogy with (4.2) for the model. From the discussion in §§2 and 5, it will be obvious that the inner solution of (8.1) will lead to the corresponding Chapman–Enskog results. We will not study here the other limits in general, but only show how the precursor results obtained for the model have to be modified for inverse power law molecules.

First, consider the dependence of the loss on the velocity for fast molecules. This can be obtained by the following general argument. Suppose the intermolecular force field is given by $F = Kr^{-m-1}$, K and m being constants and r the distance. A molecule travelling at speed v and approaching a second molecule at a value b of the impact parameter delivers to the first molecule during the ensuing collision a net impulse $I_\perp$, say, normal to its initial trajectory. This impulse is the integral of the normal force on the molecule over the time during which the force acts. The order of magnitude of the force is Kb^{-m-1} and of the time bv^{-1}; hence that of $I_\perp$ is Kb^{-m}/v. If v is large, the angle of scattering is proportional to the ratio of $I_\perp$ to the initial momentum of the molecule mv, i.e. to K/mv^2b. Hence the cross-section for a given scattering angle depends on v as

$$b^2 \sim (K/mv^2)^{2/m}.$$

The free path $\lambda = \lambda(v)$ at any velocity v is inversely proportional to the cross-section and the number density; thus

$$n\lambda \sim b^{-2} \sim (mv^2/K)^{2/m}. \tag{8.2}$$

For a Maxwell molecule $m = 4$, so $\lambda \sim v$, as in the BGK model. For rigid spheres, which correspond to $m \to \infty$, (8.2) indicates that the free path tends to a finite value even for infinite velocities, which is a well-known result (e.g. Chapman & Cowling 1952, p. 95).

The so-called Maxwell mean free path Λ is proportional to the value of λ at the mean thermal speed, so

$$n\Lambda \sim (m\beta/K)^{2/m}. \tag{8.3}$$

Thus we can write

$$L(f) \sim v/\lambda \sim \beta^{-2/m} v^{(s-4)/m}/\Lambda. \tag{8.4}$$

Now it is a general result that if the distribution is a Maxwellian F, then $G(F) = FL(F)$ for any molecular model. From (8.4) the loss is only algebraic in v, for large v; hence the behaviour of the gain in the outer limit will be dominated by that of F. It follows that, as for the model, $G(F_2)$ will be exponentially greater than $G(F_1)$ in the outer limit. It is further likely that at any point within the shock $G(f)$ will be similarly great compared to $G(F_1)$. Consequently the argument of §7.3 will be valid for a general model; in particular, far upstream of the shock we may expect to obtain the precursor by replacing G in (8.1) by the step function

$$G \simeq G(F_1) + \mathscr{H}(x)\,[G(F_2) - G(F_1)]. \tag{8.5}$$

Subsituting this and (8.4) into (8.1), we get the approximate precursor distribution

$$f(x, \mathrm{v}) \simeq F_1(\mathrm{v}) + \mathscr{H}(-v_x)\,F_2(\mathrm{v}) \exp\left\{-\frac{k|x|}{\Lambda|v_x|^{4/m}}\right\},$$

where k is some constant; equivalently, we obtain for the contracted distribution

$$f(x, v_x) \simeq F_1(v_x) + \mathscr{H}(-v_x)\, n_2(\beta_2/\pi)^{\frac{1}{2}} \exp\{-(\tilde{V}+\tilde{U}_2)^2 - X^* \tilde{V}^{-4/m}\}, \qquad (8.6)$$

where $$X^* \equiv k_1 \left(\frac{\beta_2}{\beta_1}\right)^{2/m} \frac{|x|}{\Lambda}, \quad \tilde{V} = \beta_2^{\frac{1}{2}} |v_x|, \quad \tilde{U}_2 = \beta_2^{\frac{1}{2}} u_2.$$

The moments of this distribution can be written down in terms of the functions $g_{\nu j}$ discussed in the appendix; thus, with $j = 4/m$,

$$\begin{aligned} Q^\nu - Q_1^\nu &= n_2 (\pi \beta_2^\nu)^{-\frac{1}{2}} g_{\nu j}(X^*, -U_2) \\ &\approx \frac{n_2}{(j+2)^{\frac{1}{2}} \beta_2^{\nu/2}} (\tfrac{1}{2} j X^*)^{\nu/(j+2)} \exp\left\{-\frac{j+2}{j} (\tfrac{1}{2} j X^*)^{2/(j+2)}\right\} \end{aligned} \qquad (8.7)$$

as $X^* \to \infty$. Thus, the decay of the moments is like $\exp\{-X^{*l}\}$ with the index

$$l = 2/(j+2) = m/(m+2). \qquad (8.8)$$

This index is 2/3 for Maxwell molecules, as for the model. For rigid spheres l is unity, as in continuum theories.† As most monatomic gases are represented in the range $5 < m < \infty$, we usually have $\frac{2}{3} < l < 1$.

Finally we would like to point out the possible relevance of the present results to an explanation of the ionization precursors observed in strong shocks (see, for example, Wetzel 1964). Although many complicated phenomena not considered here occur in a plasma shock, it is interesting to note that (8.8) suggests a slow decay like $\exp\{-|x|^{\frac{1}{3}}\}$ for charged particles. This could well contribute to the ionization observed at large distances ahead of a shock.

9. Conclusions

We may summarize our results here as follows. In any gas flow, no matter how small or large the nominal departure from equilibrium (as measured, say, by the conventional Knudsen number), the distribution function exhibits, in different regions in molecular velocity space, features characteristic of both collision-dominated and collisionless flow (and also, of course, of intermediate cases). There is always a collision-dominated 'inner' core at sufficiently low velocities, and a nearly collisionless 'outer' tail at sufficiently high velocities. When the nominal departure from equilibrium is small, the inner core often (but not always) covers the significant part of velocity space; when the departure is large, the outer region similarly covers most of **v**-space. The inner core is describable in terms of the local flow, but does not represent a closed description unless the departure from equilibrium is small; when it does, it is related to the Chapman–Enskog expansion, and leads to the Navier–Stokes equations (but without the help of the Euler equations as a first approximation). However, the Chapman–Enskog and the inner expansions are never uniformly valid in **v**-space. For a weak shock it proves necessary to construct a third, intermediate limit which links

† The value of l is largely determined by the form of the loss term; hence, at least in situations of the kind we are considering here, the assumption on the gain term in the BGK model is not as restrictive as it may seem at first sight.

the inner and outer solutions. The intermediate and outer solutions, which together describe the distribution of 'fast' particles, depend in general on integrals over the flow, and so are global, in contrast to the inner expansion which is local. However, in a weak shock, these solutions for the fast particles also depend only on the local flow for molecules going downstream; in fact, at sufficiently high velocities, molecules travelling from the colder to the hotter parts of the flow accumulate by streaming, with negligible attenuation. But those going from the hotter to colder parts attenuate by collisions, with negligible additional gain by scattering. It is these latter particles which act as a precursor from the hot side of the shock in the unshocked gas (or the cold side). The departure from equilibrium on the cold side is eventually dominated by this precursor, i.e. by the outer solution, rather than by higher order terms of the inner solution.

Thus, the non-uniformity of the inner solution in $\mathbf{v}$-space results in the non-uniformity of the Navier–Stokes and other continuum solutions in $\mathbf{x}$-space. This can be thought of as an extension of a result familiar in singular perturbation problems involving differential equations; namely that an approximate solution valid everywhere must be available before the solution can be improved *anywhere*. The extension is that in kinetic theory (and in similar problems involving integral equations with a parameter like $\mathbf{v}$), a solution for the distribution f uniformly valid in $\mathbf{v}$ may be a pre-requisite not only for improving f anywhere, but also for obtaining the moments of f uniformly in $\mathbf{x}$. Stated *in these* terms, it will be seen that the failure of the higher order terms of the Chapman–Enskog expansion to effect any appreciable improvement over the Navier–Stokes solution bears some mathematical resemblance to the failure of the Stokes expansions in low Reynolds number flow (e.g. Kaplun & Lagerstrom 1957). Further work to elucidate this idea is now in progress.

One specific result from this kind of analysis is the prediction of the mode of decay in the precursor of a shock. The softer the intermolecular potential, the slower is the decay. The BGK model predicts a decay which is also characteristic of Maxwell molecules. As most monatomic gases possess harder potentials than the Maxwell molecule, the decay will in general be somewhat faster. Only for a hard sphere gas is this decay like $e^{-|x|}$, as predicted by continuum theories.

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Appendix

We consider here a class of integrals defined by

$$g_{\nu j}(X^*, U) \equiv \int_0^\infty V^\nu \exp\{-(V-U)^2 - X^* V^{-j}\}\, dV, \tag{A 1}$$

where $X^* > 0$. The particular case $j = 1$ has been studied by Chahine & Narasimha (1964), who show that for sufficiently large X^* (namely $X^* \gg U^3$), the leading term in an asymptotic expansion of $g_{\nu 1} \equiv g_\nu$ is given by

$$g_\nu(X^*, U) \approx (\tfrac{1}{3}\pi)^{\frac{1}{2}} (\tfrac{1}{2}X^*)^{\nu/3} \exp\{-3(\tfrac{1}{2}X^*)^{\frac{2}{3}}\} \quad (X^* \to \infty). \tag{A 2}$$

The same reference contains other results and brief tables.

If $j = 0$, the term containing X^* can be taken outside the integral, and it is easily shown that

$$g_{\nu 0}(X^*, U) = \tfrac{1}{2} e^{-X^*} \sum_{i=0}^{\nu} \binom{\nu}{i} U^{\nu-i} \left[\Gamma\left(\frac{i+1}{2}\right) - (-)^i \Gamma\left(\frac{i+1}{2}, U^2\right)\right], \tag{A 3}$$

where $\Gamma(\alpha, x)$ is defined by (7.5a).

For intermediate values of j, an asymptotic development for large X^* can be easily obtained as follows. The saddle point of the integrand is located at the root V_0 of the equation

$$2V_0^{j+1}(V_0 - U) = jX^*,$$

which for large X^* is given by

$$V_0 = (\tfrac{1}{2}jX^*)^{1/(j+2)} + O(U).$$

Integrating around the saddle point as usual, we obtain

$$g_{j\nu}(X^*, U) \approx \left(\frac{\pi}{j+2}\right)^{\frac{1}{2}} (\tfrac{1}{2}jX^*)^{\nu/(j+2)} \exp\left\{-\frac{j+2}{j} (\tfrac{1}{2}jX^*)^{2/(j+2)}\right\}. \tag{A 4}$$

For $j = 1$ this reduces to (A 2).

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